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FAST TRACK COMMUNICATION

Non-skew-symmetric classical r-matrices and integrable cases of the reduced BCS model

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Abstract

We consider generalized quantum Gaudin systems in an external magnetic field associated with non-skew-symmetric $sl(2)$-valued classical r-matrices. We calculate spectra of the generating function of the corresponding Hamiltonians using the algebraic Bethe ansatz. We apply these results to the construction of integrable fermionic Hamiltonians of a generalized BCS type. We investigate the special cases when the corresponding integrable Hamiltonians contain only a pairing interaction term and consider an example of such a situation associated with a special non-skew-symmetric r-matrix.

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1. Introduction

During the last decade, new interest has arisen in the BCS (Bardeen, Cooper, Schrieffer) model [1] or rather in its 'reduced' finite-fermion version characterized by the following Hamiltonian:

$$\hat{H}_{\mathrm{RBCS}} = \sum_{l=1}^{N} \epsilon_l \left(c^{\dagger}_{l,+}c_{l,+} + c^{\dagger}_{l,-}c_{l,-}\right) - \sum_{m,l=1}^{N} g_{ml} c^{\dagger}_{m,+}c^{\dagger}_{m,-}c_{l,-}c_{l,+}, \tag{1}$$

where g_{ml} are coupling constants, $c^{\dagger}_{m,+}$, $c_{l,+}$, $c^{\dagger}_{m,-}$, $c_{l,-}$ are the fermion creation–anihilation operators corresponding to the two (time-reversed) states labeled by the energies ϵ_m and indices $+, -$ (spins), N is the number of pairs of fermions.

In the case of 'equal strength' or uniform coupling i.e. when $g_{ml} = g$, $\forall l, m \in \overline{1, N}$ the reduced BCS Hamiltonian was shown by Richardson [2, 3] to be exactly solvable. Recently, it has been shown [6] that its exact solvability is a consequence of its *complete quantum integrability*. It turned out that after introducing the 'pseudo-spin' operators the reduced BCS Hamiltonian of Richardson can be expressed as a function of the 'rational Gaudin spin–chain

Hamiltonians in an external magnetic field' [4, 5] corresponding to the classical $sl(2)$-valued rational r-matrix and the representation with the highest weight $\lambda = \frac{1}{2}$ for all spins in the chain.

The interpretation of the Hamiltonian of Richardson in terms of the rational Gaudin model gives a clue for the construction of its integrable generalization. In such a way, in previous work [7–10] an integrable BCS-type Hamiltonian with non-uniform coupling constants was constructed using the trigonometric Gaudin model based on the skew-symmetric trigonometric r-matrix. This 'modified' Hamiltonian has the following form:

$$\hat{H}_{\text{GBCS}} = \sum_{l=1}^{N} \epsilon_l (c_{l,+}^{\dagger} c_{l,+} + c_{l,-}^{\dagger} c_{l,-}) + \sum_{m,l=1}^{N} g_{ml} c_{m,+}^{\dagger} c_{m,-}^{\dagger} c_{l,-} c_{l,+}$$
$$+ \sum_{m,l=1}^{N} U_{ml} \sum_{\sigma,\sigma' \in +,-} c_{m,\sigma}^{\dagger} c_{m,\sigma} c_{l,\sigma'}^{\dagger} c_{l,\sigma'}, \qquad (2)$$

where non-zero coefficients g_{ml}, U_{ml} are not arbitrary but depend on the matrix elements of the trigonometric r-matrix (see also [11] for a review). The same Hamiltonian may be obtained using a limit of the trigonometric quantum R-matrix and XXY model [12]. In an analogous way, the standard Richarson's Hamiltonian is recovered using a limit of the quantum rational R-matrix and XXX model [13]. This is explained by the fact that skew-symmetric classical r-matrices and the corresponding integrable models are obtained from the quantum R-matrices and respective integrable models by the quasi-classical limit. In this context, it is necessary to note that there exists a special one-parametric family of integrable deformations of the Richardson's Hamiltonian also having the reduced BCS form (1) and containing Richarson's Hamiltonian as a limiting case. It is the so-called 'russian doll BCS model' connected not with the quasiclassical limit of a quantum rational R-matrix but with a quantum rational R-matrix itself [14].

In our previous paper [17], we generalized the result of [7–10] and obtained a more general family of integrable Hamiltonians of type (2). Our approach was based not on the models associated with quantum groups and not on their 'quasiclassical' counterparts—ordinary Gaudin models, but on the so-called 'generalized' Gaudin models in an external magnetic field [16]. Contrary to ordinary Gaudin models, our generalized Gaudin models are based on *non-skew-symmetric* classical r-matrices instead of skew-symmetric ones. Non-skew-symmetric r-matrices satisfy a 'generalized' classical Yang–Baxter equation instead of the ordinary classical Yang–Baxter equation and are not in general connected with quantum groups or related structures.

In this communication, we investigate special integrable fermionic Hamiltonians (2) that are obtained from the generalized Gaudin spin systems in an external magnetic field. The purpose of our investigation is to construct new integrable cases of the Hamiltonian (1), i.e. to find the cases when the coefficients U_{ml} in the Hamiltonians (2) are equal to zero and the corresponding Hamiltonian has only pairing interaction term. The Hamiltonian with pairing interaction (1) is more widely used in physics, in particular nuclear physics, than the 'modified' Hamiltonian (2). That is why the problem of its construction is physically important. We propose a simple method of getting rid of the third summand in the integrable Hamiltonian (2) in order to obtain a Hamiltonian with pairing interaction only. Our technique is based on the algebra of $sl(2)$-valued quantum Lax operators $\hat{L}(u)$ satisfying the linear r-matrix bracket. Integrable Hamiltonians of type (1) are constructed as the coefficients multiplying a pole of the fixed order of the generating function of quantum integrals $\hat{\tau}(u) = \text{tr}(\hat{L}(u))^2$ in a specially chosen point. We apply the proposed method to the special non-skew-symmetric r-matrix of the following explicit from:

$$r^c_{12}(u, v) = \left(\frac{v^2}{u^2 - v^2} + c\right) X_3 \otimes X_3 + \frac{uv}{2(u^2 - v^2)}(X_+ \otimes X_- + X_- \otimes X_+), \tag{3}$$

where c is an arbitrary constant, which is the simplest generalization of the skew-symmetric trigonometric r-matrix and coincides with it in the special partial case $c = \frac{1}{2}$.

It turned out that for the case of the r-matrix (3) only in the case $c = 1$ one can get rid of the third summand in the Hamiltonian (2) and obtain the integrable Hamiltonian of the form (1). In this case, we obtain the following integrable BCS-type Hamiltonian:

$$\hat{H}_{\mathrm{GBCS}} = \sum_{l=1}^{N} \epsilon_l (c^{\dagger}_{l,+} c_{l,+} + c^{\dagger}_{l,-} c_{l,-}) - g \sum_{m,l=1}^{N} \sqrt{\epsilon_m \epsilon_l} c^{\dagger}_{m,+} c^{\dagger}_{m,-} c_{l,-} c_{l,+}. \tag{4}$$

Note that contrary to the Hamiltonian of Richardson, Hamiltonian (4) has a non-uniform 'factorized strength' coupling.

We diagonalize the constructed Hamiltonians by means of the algebraic Bethe ansatz technique. We show that the Hamiltonian (4) has the following eigenvalues:

$$h_{\mathrm{GBCS}} = 2\left(\sum_{i=1}^{M} E_i\right),$$

where E_i are the solutions of the Bethe-type equations:

$$\frac{1}{2}\sum_{k=1}^{N} \frac{\epsilon_k}{\epsilon_k - E_i} - \sum_{j=1, j\neq i}^{M} \frac{E_j}{E_j - E_i} = \frac{1}{g}, \qquad i \in 1, \ldots, M.$$

The structure of this communication is as follows. In section 2, we describe the general algebraic approach to Gaudin-type models based on non-skew-symmetric classical r-matrices. In section 3, we describe the algebraic Bethe ansatz for this case. In section 4, we describe a general procedure of a construction of the fermionic Hamiltonian (1) with the help of Gaudin-type models. At last, in section 5 we obtain and diagonalize the Hamiltonian (4).

2. Quantum integrable systems and classical *r*-matrices

2.1. General classical r-matrices and 'shift elements'

Let $\mathfrak{g} = sl(2)$ be the Lie algebra of traceless 2×2 matrices over the field of complex numbers. Let $\{X_3, X_+, X_-\}$, be the root basis in $sl(2)$ with the commutation relations

$$[X_3, X_\pm] = \pm X_\pm, [X_+, X_-] = 2X_3.$$

Definition 1. *A function of two complex variables $r(u_1, u_2)$ with values in the tensor square of the algebra $sl(2)$ is called a classical r-matrix if it satisfies the following 'generalized' classical Yang–Baxter equation [19],[20],[21]:*

$$[r_{12}(u_1, u_2), r_{13}(u_1, u_3)] = [r_{23}(u_2, u_3), r_{12}(u_1, u_2)] - [r_{32}(u_3, u_2), r_{13}(u_1, u_3)], \tag{5}$$

where $r_{12}(u_1, u_2) \equiv \sum_{\alpha,\beta=1}^{3} r^{\alpha\beta}(u_1, u_2) X_\alpha \otimes X_\beta \otimes 1$, $r_{13}(u_1, u_3) \equiv \sum_{\alpha,\beta=1}^{3} r^{\alpha\beta}(u_1, u_3) X_\alpha \otimes 1 \otimes X_\beta$, $r_{23}(u_2, u_3) \equiv \sum_{\alpha,\beta=1}^{3} r^{\alpha\beta}(u_2, u_3) 1 \otimes X_\alpha \otimes X_\beta$, $r_{32}(u_3, u_2) \equiv \sum_{\alpha,\beta=1}^{3} r^{\beta\alpha}(u_3, u_2) 1 \otimes X_\alpha \otimes X_\beta$ *and* $r^{\alpha\beta}(u, v)$ *are matrix elements of the r-matrix r(u,v).*

Remark 1. In the case of skew-symmetric r-matrices when $r_{12}(u_1, u_2) = -r_{21}(u_2, u_1)$, i.e. when $r^{\alpha\beta}(u_1, u_2) = -r^{\beta\alpha}(u_2, u_1)$, the generalized classical Yang–Baxter equation passes to the usual classical Yang–Baxter equation:

$$[r_{12}(u_1, u_2), r_{13}(u_1, u_3)] = [r_{23}(u_2, u_3), r_{12}(u_1, u_2) + r_{13}(u_1, u_3)]. \tag{6}$$

Let us note that contrary to the usual classical Yang–Baxter equation (6), is not possible to define quadratic Poisson structures and quantum groups as their quantization for the general solution of the generalized classical Yang–Baxter equation (5).

We will be interested only in the meromorphic r-matrices for which there exists a reparametrization $u = u(s)$, $v = v(t)$ such that the following decomposition holds true:

$$r(u(s), v(t)) = \frac{\Omega}{s - t} + r_0(u(s), v(t)), \tag{7}$$

where $r_0(u(s), v(t))$ is a holomorphic function with values in $sl(2) \otimes sl(2)$, $\Omega \in sl(2) \otimes sl(2)$ is the tensor Casimir: $\Omega = \frac{1}{2}(X_+ \otimes X_- + X_- \otimes X_+) + X_3 \otimes X_3$.

In this communication, we will consider only 'diagonal' in the root basis r-matrices of the following explicit form:

$$r(u, v) = \left(\tfrac{1}{2} r^-(u, v) X_+ \otimes X_- + \tfrac{1}{2} r^+(u, v) X_- \otimes X_+ + r^3(u, v) X_3 \otimes X_3\right). \tag{8}$$

Remark 2. Let us note that r-matrix (8) is skew-symmetric if and only if

$$r^-(u, v) = -r^+(v, u), r^3(u, v) = -r^3(v, u).$$

We will need also the following definition.

Definition 3. *A $sl(2)$-valued function of one complex variable $c(u) = c_3(u)X_3 + c_+(u)X_+ + c_-(u)X_-$ is called a 'generalized shift element' if it solves the following equation:*

$$[r_{12}(u, v), c_1(u)] - [r_{21}(v, u), c_2(v)] = 0,$$

where $c_1(u) = c(u) \otimes 1$, $c_2(v) = 1 \otimes c(v)$.

Let us explicitly construct a special 'diagonal' shift element $c(u) = c_3(u)X_3$ for the diagonal in the root basis r-matrices (8). The following proposition holds true [17].

Proposition 2.1. *For an arbitrary r-matrix of the form (8) having the regularity property (7) and constant $k \in \mathbb{C}$ the function*

$$c(u) = k c_0(u) X_3 \equiv k\left(r_0^3(u, u) - \tfrac{1}{2}(r_0^+(u, u) + r_0^-(u, u))\right) X_3, \tag{9}$$

where $r_0^\alpha(u, v)$ are regular parts of the components of the r-matrix: $r^\alpha(u(s), v(t)) = (s - t)^{-1} + r_0^\alpha(u(s), v(t))$, is a generalized shift element.

2.2. Algebra of Lax operators

Using a classical r-matrix $r(u,v)$ it is possible to define in the space of certain $sl(2)$-valued functions of u with the operator coefficients $\hat{L}(u) = \hat{L}^3(u)X_3 + \hat{L}^+(u)X_+ + \hat{L}^-(u)X_-$ the 'tensor' Lie bracket:

$$[\hat{L}_1(u), \hat{L}_2(v)] = [r_{12}(u, v), \hat{L}_1(u)] - [r_{21}(v, u), \hat{L}_2(v)], \tag{10}$$

where $\hat{L}_1(u) = \hat{L}(u) \otimes 1$, $\hat{L}_2(v) = 1 \otimes \hat{L}(v)$.

The non-trivial commutation relations (10) written in the component form are the following:

$$[\hat{L}^-(u), \hat{L}^3(v)] = -(r^3(u,v)\hat{L}^-(u) + r^-(v,u)\hat{L}^-(v)),$$

$$[\hat{L}^+(u), \hat{L}^3(v)] = (r^3(u,v)\hat{L}^+(u) + r^+(v,u)\hat{L}^+(v)),$$

$$[\hat{L}^+(u), \hat{L}^-(v)] = -\tfrac{1}{2}(r^-(u,v)\hat{L}^3(u) + r^+(v,u)\hat{L}^3(v)).$$

The components of the Lax operator $\hat{L}^\alpha(u)$ depend on an auxiliary parameter u and the non-commuting quantum dynamical variables. The following proposition is true [16].

Proposition 2.2. *Let $\hat{S}^i_+, \hat{S}^i_-, \hat{S}^i_3, i = 1, \ldots, N$ be linear operators in some Hilbert space that constitute a Lie algebra isomorphic to $so(3)^{\oplus N} \simeq sl(2)^{\oplus N}$ with the commutation relations:*

$$[\hat{S}^i_+, \hat{S}^j_-] = 2\delta^{ij}\hat{S}^j_3,\ [\hat{S}^i_+, \hat{S}^j_3] = -\delta^{ij}\hat{S}^j_+,\ [\hat{S}^i_-, \hat{S}^j_3] = \delta^{ij}\hat{S}^j_-, \tag{11}$$

$$[\hat{S}^i_+, \hat{S}^j_+] = [\hat{S}^i_-, \hat{S}^j_-] = [\hat{S}^i_3, \hat{S}^j_3] = 0. \tag{12}$$

Let $\nu_k,\ \nu_k \neq \nu_l,\ k,l = 1, \ldots, N$ be some fixed points on the complex plane belonging to the open region U in which the r-matrix r(u,v) possesses the decomposition (7). Let $c(u) = c_3(u)X_3 + c_+(u)X_+ + c_-(u)X_-$ be a shift element. Then the quantum Lax operator with the following components,

$$\hat{L}^3(u) = \sum_{k=1}^{N} r^3(\nu_k, u)\hat{S}^k_3 + c_3(u),\ \hat{L}^\pm(u) = \frac{1}{2}\sum_{k=1}^{N} r^\pm(\nu_k, u)\hat{S}^k_\mp + c_\pm(u), \tag{13}$$

satisfies the commutation relations (10) with the diagonal r-matrix (8).

Remark 3. The Lax operator (13) is the Lax operator of the generalized Gaudin spin chain in an external magnetic field, where N is the number of spins in the chain and the role of the external magnetic field is played by a generalized shift element $c(u)$ (see [16]).

2.3. Quantum integrals

In this section, we will explain the connection of classical non-skew-symmetric r-matrices with quantum integrability. It was shown in our previous paper [15] that just like in the case of classical r-matrix Lie–Poisson brackets [18],[19],[20] the Lie bracket (10) leads to an algebra of mutually commuting quantum integrals.

Let us consider the following quadratic in generators of the Lax algebra operators:

$$\hat{\tau}(u) = (\hat{L}^3(u))^2 + 2(\hat{L}^+(u)\hat{L}^-(u) + \hat{L}^-(u)\hat{L}^+(u)). \tag{14}$$

In order to obtain quantum integrable systems, one has to show that $[\hat{\tau}(u), \hat{\tau}(v)] = 0$. This equality does not follow directly from the classical Poisson commutativity of $\tau(u)$ and $\tau(v)$ with respect to the corresponding Lie–Poisson brackets due to the problem of ordering of quantum operators. Nevertheless, the following theorem holds true [15].

Theorem 2.1. *Let $\hat{L}(u)$ be the Lax operator satisfying the commutation relations (10). Assume that in some open region $U \times U \subset \mathbb{C}^2$ the function r(u,v) is meromorphic and possesses the decomposition (7). Then the operator-valued function $\hat{\tau}(u)$ is a generator of a commutative algebra, i.e.:*

$$[\hat{\tau}(u), \hat{\tau}(v)] = 0.$$

The generating function of the quantum integrals of the generalized spin chain in a magnetic field in the case of diagonal shift elements (i.e. when $c_\pm(u) = 0$) has the following explicit form:

$$\hat{\tau}(u) = \left(\sum_{l=1}^{N} 2c_3(u) r^3(\nu_l, u) S_3^{(l)} + \frac{1}{2} \sum_{k,l=1}^{N} r^-(\nu_k, u) r^+(\nu_l, u) (\hat{S}_+^{(k)} \hat{S}_-^{(l)} + \hat{S}_-^{(k)} \hat{S}_+^{(l)}) \right.$$
$$\left. + \sum_{k,l=1}^{N} r^3(\nu_k, u) r^3(\nu_l, u) \hat{S}_3^{(k)} \hat{S}_3^{(l)} + (c_3(u))^2 \right).$$

In the following section, we will diagonalize this generating function by means of Bethe ansatz.

3. Diagonalization of quantum Hamiltonians

Let us consider a finite-dimensional irreducible representation of the algebra $sl(2)^{\oplus N}$ in some space $\mathcal{H}$. Due to the fact that any irreducible representation of the direct sum of the Lie algebras is a tensor product of irreducible representations of their components, we will have $\mathcal{H} = V^{\lambda_1} \otimes V^{\lambda_2} \otimes \cdots \otimes V^{\lambda_N}$, where V^{λ_k} is an irreducible finite-dimensional representation of the kth copy of $sl(2)$ with the spin λ_k, where $\lambda_k \in \frac{1}{2}\mathbb{N}$.

Each representation V^{λ_k} contains the highest weight vector v_{λ_k} such that

$$\hat{S}_+^k \mathrm{v}_{\lambda_k} = 0, \quad \hat{S}_3^k \mathrm{v}_{\lambda_k} = \lambda_k \mathrm{v}_{\lambda_k}, \tag{15}$$

and the whole space V^{λ_k} is spanned by $\mathrm{v}_{\lambda_k}^m = (\hat{S}_-^k)^m \mathrm{v}_{\lambda_k}$, $m \in 0, \ldots, 2\lambda_k$.

The Casimir function $\hat{C}_2^k$ acts on each vector $\mathrm{v}_{\lambda_k}^m \in V^{\lambda_k}$ in the usual way:

$$\hat{C}_2^k \mathrm{v}_{\lambda_k}^m = \lambda_k(\lambda_k + 1) \mathrm{v}_{\lambda_k}^m.$$

Let us consider the following 'vacuum' vector in the space $\mathcal{H}$: $|0\rangle = \mathrm{v}_{\lambda_1} \otimes \mathrm{v}_{\lambda_2} \otimes \cdots \otimes \mathrm{v}_{\lambda_N}$. We have that $\hat{L}^-(u)|0\rangle = 0$, due to the definition of $\hat{L}^-(u)$ and the equality (15). It is also easy to show that the vector $|0\rangle$ is an eigenvector for the generating function of the quantum Hamiltonians:

$$\hat{\tau}(u)|0\rangle = \left(\Lambda_3(u)^2 + \partial_u \Lambda_3(u) + (2c_3(u) + r_0^-(u,u) + r_0^+(u,u))\Lambda_3(u) + c_3^2(u)\right)|0\rangle,$$

where $\Lambda_3(u) = \sum_{k=1}^{N} r^3(\nu_k, u)\lambda_k$ and we have used that

$$[\hat{L}^+(u), \hat{L}^-(u)] = -\tfrac{1}{2}(\partial_u \hat{L}^3(u) + (r_0^-(u,u) + r_0^+(u,u))\hat{L}^3(u)).$$

Let us now construct other eigenvectors of $\hat{\tau}(u)$ using the Bethe ansatz technique.

The following theorem holds true [17].

Theorem 3.1. *Let us consider the following Bethe-type vectors:*

$$|v_1 v_2 \cdots v_M\rangle = \hat{L}^+(v_1)\hat{L}^+(v_2)\cdots\hat{L}^+(v_M)|0\rangle, \tag{16}$$

where the complex parameters v_i satisfy the following Bethe-type equations:

$$\sum_{k=1}^{N} r^3(\nu_k, v_i)\lambda_k - \sum_{j=1, j\neq i}^{M} r^3(v_j, v_i) = c_0(v_i) - c_3(v_i), \qquad i \in 1, \ldots, M, \tag{17}$$

$c_0(v) = r_0^3(v,v) - \frac{1}{2}(r_0^+(v,v) + r_0^-(v,v))$ *and* $c_3(v)$ *is a shift function.*

Then the vectors $|v_1 v_2 \cdots v_M\rangle$ are the eigenvectors of the generating function of the quantum Hamiltonians $\hat{\tau}(u)$: $\hat{\tau}(u)|v_1 v_2 \cdots v_M\rangle = \Lambda(u|\{v_i\})|v_1 v_2 \cdots v_M\rangle$ with the following eigenvalues:

$$\Lambda(u|\{v_i\}) = \Big(\Lambda_3(u) - \sum_{i=1}^{M} r^3(v_i, u)\Big)^2 - \sum_{i=1}^{M} r^+(v_i, u) r^-(v_i, u) + \partial_u \Lambda_3(u)$$
$$+ (r_0^-(u, u) + r_0^+(u, u))\Lambda_3(u) + 2c_3(u)\Big(\Lambda_3(u) - \sum_{i=1}^{M} r_3(v_i, u)\Big) + c_3^2(u), \qquad \textit{where} \quad \Lambda_3(u) = \sum_{k=1}^{N} r^3(\nu_k, u)\lambda_k. \tag{18}$$

4. Integrable BCS-type models and *r*-matrices

4.1. Fermionization

Having obtained the quantum integrable spin system, it is possible to derive, using them, integrable fermionic systems. For this purpose, it is necessary to consider the realization of the corresponding spin operators in terms of fermionic creation–anihilation operators.

Let us consider the fermionic creation–anihilation operators $c^{\dagger}_{j,\sigma'}, c_{i,\sigma}$, $i, j \in \overline{1, N}$, $\sigma, \sigma' \in \{+, -\}$ with the following anti-commutation relations:

$$c^{\dagger}_{i,\sigma} c_{j,\sigma'} + c_{j,\sigma'} c^{\dagger}_{i,\sigma} = \delta_{\sigma\sigma'}\delta_{ij}, \quad c^{\dagger}_{i,\sigma} c^{\dagger}_{j,\sigma'} + c^{\dagger}_{j,\sigma'} c^{\dagger}_{i,\sigma} = 0, \quad c_{i,\sigma} c_{j,\sigma'} + c_{j,\sigma'} c_{i,\sigma} = 0.$$

Then the following formulae,

$$\hat{S}^j_+ = c_{j,-} c_{j,+}, \; \hat{S}^j_- = c^{\dagger}_{j,+} c^{\dagger}_{j,-}, \; \hat{S}^j_3 = \frac{1}{2}(1 - c^{\dagger}_{j,+} c_{j,+} - c^{\dagger}_{j,-} c_{j,-}), \; i, j \in 1, N, \tag{19}$$

provide the realization of the Lie algebra $sl(2)^{\oplus N}$ with the highest weight $\lambda_1 = \lambda_2 = \cdots = \lambda_N = \frac{1}{2}$.

Remark 4. Let us note that in such a realization in a representation of $sl(2)^{\oplus N}$ with a highest vectors such that $\hat{S}^j_+|0>= 0$ the operators $c_{j,\pm}$ play the role of 'anihilation operators' and $c^{\dagger}_{j,\pm}$ play the role of creation operators.

4.2. Special BCS-type Hamiltonians

Now, let us obtain integrable fermionic Hamiltonians using the realization (19) and the constructed in the previous sections integrable spin chains in a magnetic field. Let $\hat{\tau}(u)$ be the generating function of the quantum integrals of the generalized spin chain in a magnetic field. Let us fix some additional point ν_0 and consider the following Hamiltonian:

$$\hat{H}^{(2)}_{\nu_0} = \tfrac{1}{2}\mathrm{res}_{\mu(u)=\mu(\nu_0)} \hat{\tau}(u),$$

where μ is some specially chosen new 'spectral parameter' $\mu = \mu(u)$, $\mu(\nu_0) \neq \mu(\nu_k)$, $k \in 1, N$. Let us assume that the point $\mu_0 = \mu(\nu_0)$ is taken in such a way that this Hamiltonian is not trivial. By a direct calculation we obtain its following explicit form:

$$\hat{H}^{(2)}_{\nu_0} = \sum_{l=1}^{N} \epsilon_l(\nu_0)\hat{S}^{(l)}_3 + \sum_{k,l=1}^{N} g_{kl}(\nu_0)\hat{S}^{(k)}_- \hat{S}^{(l)}_+ + \sum_{k,l=1}^{N} U_{kl}(\nu_0)\hat{S}^{(k)}_3 \hat{S}^{(l)}_3 + E_0(\nu_0), \tag{20}$$

$$\epsilon_l(\nu_0) = \mathrm{res}_{\mu(u)=\mu(\nu_0)}\big(c_3(u) r^3(\nu_l, u) + \tfrac{1}{2} r^-(\nu_l, u) r^+(\nu_l, u)\big),$$
$$g_{kl}(\nu_0) = \tfrac{1}{4}\mathrm{res}_{\mu(u)=\mu(\nu_0)}(r^-(\nu_k, u) r^+(\nu_l, u) + r^-(\nu_l, u) r^+(\nu_k, u)),$$
$$U_{kl}(\nu_0) = \tfrac{1}{2}\mathrm{res}_{\mu(u)=\mu(\nu_0)}(r^3(\nu_k, u) r^3(\nu_l, u)), \; E_0(\nu_0) = \tfrac{1}{2}\mathrm{res}_{\mu(u)=\mu(\nu_0)}(c^3(u))^2.$$

In the case of a special choice of the points ν_0 the Hamiltonian (20) may be simplified not to contain its third term. In more details, let the special point $\mu(u) = \mu(\nu_0)$, $\mu(\nu_0) \neq \mu(\nu_k)$, $k \in \overline{1, N}$ be such that the following condition is satisfied:

$$U_{kl}(\nu_0) = \frac{1}{2}\mathrm{res}_{\mu(u)=\mu(\nu_0)}(r^3(\nu_k, u)r^3(\nu_l, u)) = 0. \tag{21}$$

The Hamiltonian $\hat{H}^{(2)}_{\nu_0}$ associated with a diagonal r-matrix and the point ν_0 in which the condition (21) is satisfied has the form

$$\hat{H}^{(2)}_{\nu_0} = \sum_{l=1}^{N} \epsilon_l(\nu_0)\hat{S}_3^{(l)} + \sum_{k,l=1}^{N} g_{kl}(\nu_0)\hat{S}_-^{(k)}\hat{S}_+^{(l)} + E_0(\nu_0). \tag{22}$$

In the most important case when $\lambda_1 = \cdots = \lambda_N = \frac{1}{2}$ we obtain the following Hamiltonian of the BCS type written in terms of fermionic operators ($\hat{H}_{\mathrm{GBCS}} \equiv \hat{H}^{(2)}_{\nu_0} - E_0(\nu_0) - \frac{1}{2}\sum_{l=1}^{N}\epsilon_l(\nu_0)$):

$$\hat{H}_{\mathrm{GBCS}} = -\frac{1}{2}\sum_{l=1}^{N}\epsilon_l(\nu_0)(c^\dagger_{l,+}c_{l,+} + c^\dagger_{l,-}c_{l,-}) + \sum_{m,l=1}^{N} g_{ml}(\nu_0)c^\dagger_{m,+}c^\dagger_{m,-}c_{l,-}c_{l,+}. \tag{23}$$

5. Example

In this section, we will explicitly obtain a new example of the integrable BCS-type model with non-uniform coupling constants associated with a special *non-skew-symmetric* r-matrix.

5.1. 'Shifted' non-skew-symmetric classical r-matrices

Let us consider the non-skew-symmetric solution of the generalized classical Yang–Baxter equation on $sl(2)$ of the following explicit form:

$$r^c_{12}(u, v) = \left(\frac{v^2}{u^2 - v^2} + c\right) X_3 \otimes X_3 + \frac{uv}{2(u^2 - v^2)}(X_+ \otimes X_- + X_- \otimes X_+). \tag{24}$$

It is possible to show (see [17]) that it satisfies the generalized classical Yang–Baxter equation for an arbitrary value of constant $c \in \mathbb{C}$.

The components of the r-matrix (24) are $r^{c,3}(u, v) = \dfrac{v^2}{u^2 - v^2} + c$, $r^{c,\pm}(u, v) = \dfrac{uv}{u^2 - v^2}$. The r-matrix (24) is not in general skew-symmetric: if $c \neq \frac{1}{2}$ then $r^{c,3}(u, v) \neq -r^{c,3}(v, u)$.

The parametrization in which the r-matrix (24) possesses the decomposition (7) is the 'hyperbolic' parametrization: $u^2 = e^s$, $v^2 = e^t$. For such a parametrization we have

$$r_{12}(u(s), v(t)) = \frac{1}{s - t}X_3 \otimes X_3 + \frac{1}{2(s - t)}(X_+ \otimes X_- + X_- \otimes X_+) + r^0_{12}(s - t).$$

Using this parametrization it is possible to show that $r^{c,3}_0(u, u) = c - \frac{1}{2}$, $r^{c,\pm}_0(u, u) = 0$.

Using this, formula (9) one obtains that diagonal shift element $c(u) = c_3(u)X_3$ has the form $c(u) = k(c - \frac{1}{2})X_3$, $k \in \mathbb{C}$, i.e. in this case one can put simply $c_3(u) = \mathrm{const} \equiv c'$.

5.2. Special BCS-type Hamiltonians

Let us consider the case of the special BCS-type Hamiltonians that correspond to the classical r-matrix $r^c_{12}(u, v)$. The generating function $\hat{\tau}(u)$ is an even function of the spectral parameter that is why one has to calculate its residues with respect to the special point μ_0 of the spectral parameter $\mu(u) = u^2$. We will take, for example, the point $\mu_0 = \nu_0^2 = 0$.

Decomposing the functions $r^{c,3}(\nu_k, u)$, $r^{c,+}(\nu_k, u) r^{c,-}(\nu_l, u)$ with respect to u^{-2} and calculating the coefficients by u^{-2} in these decompositions, we obtain that

$$U_{kl}(0) = \frac{1}{2}\mathrm{res}_{u^2=0} r^{c,3}(\nu_k, u) r^{c,3}(\nu_l, u) = -\frac{(c-1)}{2}\left(\nu_k^2 + \nu_l^2\right),$$

$$g_{kl}(0) = \frac{1}{2}\mathrm{res}_{u^2=0} r^{c,+}(\nu_k, u) r^{c,-}(\nu_l, u) = \frac{\nu_k \nu_l}{2},$$

$$\epsilon_l(0) = \mathrm{res}_{u^2=0}(c_3(u) r^{c,3}(\nu_l, u) + \frac{1}{2} r^{c,-}(\nu_l, u) r^{c,+}(\nu_l, u)) = -\left(c' - \frac{1}{2}\right)\nu_l^2,$$

which yields the following BCS-type Hamiltonian (20) calculated in the point $\nu_0 = 0$:

$$\hat{H}_0^{(2)} = -\left(c' - \frac{1}{2}\right)\sum_{l=1}^{N}\nu_l^2 \hat{S}_3^{(l)} + \frac{1}{2}\sum_{k,l=1}^{N}\nu_k\nu_l \hat{S}_-^{(k)}\hat{S}_+^{(l)} - \frac{1}{2}(c-1)\sum_{k,l=1}^{N}\left(\nu_k^2 + \nu_l^2\right)\hat{S}_3^{(k)}\hat{S}_3^{(l)}. \quad (25)$$

The obtained Hamiltonian depends on two parameters c' and c. First of these parameters departs from the shift element and is interpreted as an external magnetic field. Second departs from the r-matrix itself and measures its deviation from skew-symmetry. Only under the special choice of the second parameter, namely $c = 1$, one can get rid of the unwanted third term in the Hamiltonian (25). In this case, it acquires the following form:

$$\hat{H}_0^{(2)} = -\left(c' - \frac{1}{2}\right)\sum_{l=1}^{N}\nu_l^2 \hat{S}_3^{(l)} + \frac{1}{2}\sum_{k,l=1}^{N}\nu_k\nu_l \hat{S}_-^{(k)}\hat{S}_+^{(l)}. \quad (26)$$

In the irreducible representations with $\lambda_k = \frac{1}{2}, k \in \overline{1, N}$ introducing notations $\epsilon_l \equiv \nu_l^2$, $g \equiv (\frac{1}{2} - c')^{-1}$, multiplying $\hat{H}_0^{(2)}$ by $(-2g)$ one obtains the following integrable BCS Hamiltonian:

$$\hat{H}_{\mathrm{GBCS}} = \sum_{l=1}^{N}\epsilon_l\left(c_{l,+}^{\dagger}c_{l,+} + c_{l,-}^{\dagger}c_{l,-}\right) - g\sum_{m,l=1}^{N}\sqrt{\epsilon_m\epsilon_l}\, c_{m,+}^{\dagger}c_{m,-}^{\dagger}c_{l,-}c_{l,+}. \quad (27)$$

5.3. Spectrum and Bethe equations

Let us explicitly find the Bethe equations and spectrum for the obtained Hamiltonian (27). Using the explicit form of the classical r-matrix $r_{12}^{c}(u, v)$ for $c = 1$ we obtain the following explicit expression for the Bethe equations (17):

$$\sum_{k=1}^{N}\frac{\nu_k^2\lambda_k}{\nu_k^2 - v_i^2} - \sum_{j=1, j\neq i}^{M}\frac{v_j^2}{v_j^2 - v_i^2} = \left(\frac{1}{2} - c'\right) \equiv \frac{1}{g}, \; i \in 1, \ldots, M. \quad (28)$$

Taking the residue in the point $u^2 = 0$ of the general expression (18) and taking into account the explicit form of the r-matrix one obtains the following answer for the spectrum of the Hamiltonian $\hat{H}_0^{(2)}$: $h_0^{(2)} = \left(c' - \frac{1}{2}\right)\left(\sum_{i=1}^{M} v_i^2 - \sum_{k=1}^{N}\nu_k^2\lambda_k\right)$.

In a new notation, we get the following answer for spectrum of the Hamiltonian $\hat{H}_{\mathrm{GBCS}}$: $h_{\mathrm{GBCS}} = 2\left(\sum_{i=1}^{M} E_i\right)$, where $E_i \equiv v_i^2, \epsilon_k \equiv \nu_k^2, g \equiv \left(\frac{1}{2} - c'\right)^{-1}$. The Bethe equations (28) are written in the new notations as follows:

$$\frac{1}{2}\sum_{k=1}^{N}\frac{\epsilon_k}{\epsilon_k - E_i} - \sum_{j=1, j\neq i}^{M}\frac{E_j}{E_j - E_i} = \frac{1}{g}, \qquad i \in 1, \ldots, M. \quad (29)$$

We have used that in the case of fermionic realization (19) we have $\lambda_k = \frac{1}{2}, k \in \overline{1, N}$.

6. Conclusion and discussion

In this short communication, we have constructed an integrable case of the reduced BCS Hamiltonian consisting of kinetic and pairing interaction terms. The constructed Hamiltonian possesses the 'factorized strength coupling'. We hope that our reduced BCS Hamiltonian will give a better approximation to real physical Hamiltonians (for example in nuclear physics) than the traditional 'equal strength coupling' Hamiltonian of Richardson. In the context of possible applications, it is also necessary to mention recent paper [22] where a similar fermionic Hamiltonian was considered in the context of the so-called $p_x + ip_y$ model of superconductors.

It will be very interesting to construct correlation functions for the obtained model. For this purpose, it is necessary to prolong the technique of Sklyanin [23] from the case of skew-symmetric r-matrices to non-skew-symmetric cases.

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